Series Cheat Sheet

This chapter introduces you to more techniques involving series. You will learn about the method of differences, finding derivatives past second order and finally how to find the Maclaurin series expansion for a given function.

Method of differences

The method of differences is essentially a trick that we can use to find the sums of some finite series. The idea is that we rewrite the general term of the series as a difference of two or more terms. This makes calculating the sum much easier as many of the terms will cancel out. To illustrate how this works in practice, lets look at the following sum:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}$$

This expression is known as the summand, or the general term.

If we were to try to sum up the series by calculating each term separately, it would look like:

$$\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\cdots+\frac{1}{n(n+1)}$$

Trying to find an expression for this sum would prove to be quite difficult. However, if we instead split the original summand by partial fractions, we have that:

$$\frac{1}{r(r+1)}=\frac{1}{r}-\frac{1}{r+1}$$

So rather than considering the original series, we can instead look at

$$\sum_{r=1}^{n} \frac{1}{r}-\frac{1}{r+1}$$

This turns out to be much easier to evaluate. To try to find this sum, we write out the first and last three terms in an effort to figure out any pattern that might exist:

$$\begin{aligned}
r=1:&\quad \frac{1}{1}-\frac{1}{2}\\
r=2:&\quad \frac{1}{2}-\frac{1}{3}\\
r=3:&\quad \frac{1}{3}-\frac{1}{4}\\
&\quad \cdots\\
r=n-2:&\quad \frac{1}{n-2}-\frac{1}{n-1}\\
r=n-1:&\quad \frac{1}{n-1}-\frac{1}{n}\\
r=n:&\quad \frac{1}{n}-\frac{1}{n+1}
\end{aligned}$$

It is important that you use the same clear vertical structure to write each term of the series, so you're able to spot the pattern. Not every pattern will be as clear as in this example.

The terms $\frac{1}{4}$ and $\frac{1}{n-2}$ will also cancel out, but with terms that are not listed.

As you can see, it turns out that most of the terms cancel out. These are colour coded. The only terms that won't cancel out in this example are the first and last terms. So we can conclude that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)}=1-\frac{1}{n+1}=\frac{n}{n+1}$$

The above process is known as the method of differences, and can be summarised into three steps:

1. Rewrite the general term of the series (i.e. summand) as a difference of two or more terms. This will either be given to you from a previous part of the question or you will be expected to use partial fractions if the general term is a fractional expression with multiple linear factors in the denominator.
2. Write out the first and last three terms of the series in the vertical format we used above. This will allow you to more easily notice which terms cancel out and which terms do not.
3. Identify all terms that do not cancel out and add them all together. Simplify your result as much as possible.

Note that we don't have to necessarily look at the first/last three terms. We could equivalently pick the first/last two if we are confident in spotting the pattern. Choosing three is advised to ensure you can still figure out the pattern for more complicated series.

Example 1: Prove that $\sum_{r=1}^{n} \frac{3}{(3r+1)(3r+4)}=\frac{an}{bn+c}$, where a, b and c are constants to be found.

First, we express the summand using partial fractions:	$\frac{3}{(3r+1)(3r+4)} \equiv \frac{A}{3r+1}+\frac{B}{3r+4}$
The working has been omitted here.	$\frac{3}{(3r+1)(3r+4)} \equiv \frac{1}{3r+1}-\frac{1}{3r+4}$
Now we write out some of the terms of the series. The terms that cancel out are matching in colour. The red terms however cancel out with another term which is not listed.	$r=1$: $\frac{1}{4}-\frac{1}{7}$ $r=2$: $\frac{1}{7}-\frac{1}{10}$ … $r=n-1$: $\frac{1}{3n-2}-\frac{1}{3n+1}$ $r=n$: $\frac{1}{3n+1}-\frac{1}{3n+4}$
Notice that the only terms that don't cancel out are the first and last terms, so our series can be expressed as:	$\therefore \sum_{r=1}^{n} \frac{3}{(3r+1)(3r+4)}=\frac{1}{4}-\frac{1}{3n+4}$
Simplifying by writing the result as one fraction:	$=\frac{3n+4-4}{4(3n+4)}=\frac{3n}{12n+16}$, so $a=3, b=12, c=16$

Higher derivatives

You also need to be comfortable with finding the higher derivatives of given functions. This is as simple as differentiating a function as many times as required. For example, to find the fourth derivative of $y=f(x)$, we must differentiate the function four times. You may need to use any techniques you learnt from Chapter 9 (Differentiation) in Pure Year 2.

- The nth derivative of $y=f(x)$ is written as $\frac{d^n y}{dx^n}$, or $f^n(x)$.

Example 2: Given that $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$, show that $(1+x^2)f'''(x)+3xf''(x)+f'(x)=0$.

We need to find the first three derivatives of $f(x)$ and show the equation is satisfied. The first derivative at first seems complicated but once we simplify it becomes much more friendly. This makes calculating further derivatives less tedious.	$f(x)=\ln\left(x+\sqrt{1+x^2}\right)$ $f'(x)=\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}=\frac{x+\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\times\frac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}$
	$f''(x)=\frac{d}{dx}\left((1+x^2)^{-\frac{1}{2}}\right)=-x(1+x^2)^{-\frac{3}{2}}$
	$f'''(x)=\frac{d}{dx}\left(-x(1+x^2)^{-\frac{3}{2}}\right)$ $=-(1+x^2)^{-\frac{3}{2}}+3x(1+x^2)^{-\frac{5}{2}}$
Now looking at the first term of the LHS of the given expression:	$(1+x^2)f'''(x)=(1+x^2)\left[-(1+x^2)^{-\frac{3}{2}}+3x^2(1+x^2)^{-\frac{5}{2}}\right]$ $=-(1+x^2)^{-\frac{1}{2}}+3x^2(1+x^2)^{-\frac{3}{2}}$
Looking at the second term:	$3xf''(x)=-3x^2(1+x^2)^{-\frac{3}{2}}$
Looking at the third term:	$f'(x)=(1+x^2)^{-\frac{1}{2}}$
Adding all three terms together gives us 0 as required:	$\therefore LHS=3x(1+x^2)^{-\frac{3}{2}}-3x^2\ (1+x^2)^{-\frac{3}{2}}+(1+x^2)^{-\frac{1}{2}}$ $-(1+x^2)^{-\frac{1}{2}}=0=RHS$

Maclaurin series

The Maclaurin series of a given function is an infinite sum of terms that estimates what the function looks like around the point $x=0$. This is a very powerful tool because some functions are very complicated and therefore difficult to analyse. For such functions, we can instead look at their Maclaurin expansions, which in many cases is much easier to work with. Remember that:

- The Maclaurin series of a given function is an approximation of that function around $x=0$.
- The more terms you find for a Maclaurin series, the better the approximation becomes.

To find the Maclaurin series expansion for a function $f(x)$, you can use the following formula:

- $f(x)=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+\cdots+f^r(0)\frac{x^r}{r!}+\cdots$

However, there is a catch: the Maclaurin series for a given function is only valid for values of x that cause the series to converge. The values of $f(0), f'(0), f''(0), \ldots$ must also be finite for the expansion to hold.

- The range of validity for some standard expansions will be given in the formula booklet. If you are given a compound function whose range of validity cannot be determined from the standard expansions however, you can just assume the expansion is valid for all real values of x.

Example 3: Express $\ln\cos x$ as a series in ascending powers of x up to and including the term in x^4.

To find the series expansion of $\ln\cos x$, we need to first differentiate the function four times. The chain and product rules are used here.	$f(x)=\ln\cos x$ $f'(x)=\frac{-\sin x}{\cos x}=-\tan x$ $f''(x)=-\sec^2 x$ $f'''(x)=-2\sec^2 x\tan x$ $f''''(x)=-2\sec^4 x-4\sec^2 x\tan^2 x$
Plugging $x=0$ into $f(x)$ and its derivatives:	$f(0)=\ln 1=0$ $f'(0)=-\tan(0)=0$ $f''(0)=-\sec^2(0)=-1$ $f'''(0)=-2\sec^2(0)\tan(0)=0$ $f''''(0)=-2\sec^4(0)-4\sec^2(0)\tan(0)=-2$
Now plugging our results into the formula:	$\therefore f(x)\approx 0+0\frac{x}{1!}-1\frac{x^2}{2!}+0\frac{x^3}{3!}-2\frac{x^4}{4!}$ $=\frac{-x^2}{2}-\frac{x^4}{12}$

Series expansions of compound functions

There are a set of standard Maclaurin expansions that you are given in the formulae booklet, along with their ranges of validity. You can use these to find series expansions of some compound functions, whose derivatives may be tedious to calculate or where products of functions are involved. For example, the standard expansions would be useful when finding the series expansion of $e^{\cos x}$, or $\arctan(x^2)$.

- $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots,\quad$ for all x
- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots,\quad -1<x\le 1$
- $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots,\quad$ for all x
- $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots,\quad$ for all x
- $\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots,\quad -1\le x\le 1$

These are the ranges of x for which these expansions are valid. You may need to manipulate these to find the ranges of validity for compound functions. (See example 4)

Example 4: Find the first three non-zero terms in the series expansion of $\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right)$, and state the values of x for which the expansion is valid.

We use the log division law to split up the logarithm:	$\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right)=\ln\left(\sqrt{1+2x}\right)-\ln(1-3x)$
To find the expansion of $\ln\left(\sqrt{1+2x}\right)$, we first rewrite it as $\frac{1}{2}\ln(1+2x)$	$\ln\left(\sqrt{1+2x}\right)=\frac{1}{2}\ln(1+2x)$
Then to find $\ln(1+2x)$ we use the above expansion for $\ln(1+x)$ replacing x with $2x$:	$=\frac{1}{2}\left[2x-\frac{4x^2}{2}+\frac{8x^3}{3}\right]=x-x^2+\frac{4}{3}x^3$
To find the range of values for which this expansion is valid, we use the given range above but we substitute $2x$ in place of x:	Valid for $-1<2x\le 1 \Rightarrow -\frac{1}{2}<x<\frac{1}{2}$
To find the expansion of $\ln(1-3x)$ now, we use the given expansion again but replace x with $-3x$:	$\ln(1-3x)=(-3x)-\frac{(-3x)^2}{2}+\frac{(-3x)^3}{3}$ $=-3x-\frac{9x^2}{2}-9x^3$
To find the range of values for which this expansion is valid, we use the given range above but we substitute $-3x$ in place of x:	Valid for $-1<-3x\le 1 \Rightarrow -\frac{1}{3}\le x<\frac{1}{3}$ (Signs flip when dividing by a negative quantity)
To find the required expansion now, we just subtract the expansions we found:	$\ln\left(\frac{\sqrt{1+2x}}{1-3x}\right)\approx\left[x-x^2+\frac{4}{3}x^3\right]-\left[-3x-\frac{9x^2}{2}-9x^3\right]$
The range of validity of x is simply given by where both of the ranges we found earlier are satisfied.	$\approx 4x+\frac{7}{2}x^2+\frac{31}{3}x^3,\quad -\frac{1}{3}\le x<\frac{1}{3}$